Sharp constants related to the triangle inequality in Lorentz spaces

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Abstract: We study the Lorentz spaces $L^{p,s}(R,\mu)$ in the range $1 < p < s \le \infty$, for which the standard functional

$$||f||_{p,s} = \left(\int_0^\infty (t^{1/p} f^*(t))^s \frac{dt}{t} \right)^{1/s}$$

is only a quasi-norm. We find the optimal constant in the triangle inequality for this quasi-norm, which leads us to consider the following decomposition norm:

$$||f||_{(p,s)} = \inf \left\{ \sum_k ||f_k||_{p,s} \right\},$$

where the infimum is taken over all finite representations $f = \sum_k f_k$. We also prove that the decomposition norm and the dual norm

$$||f||'_{p,s} = \sup \left\{ \int_R f g \, d\mu : ||g||_{p',s'} = 1 \right\}$$

agree for all values $p, s > 1$.

1. Introduction

The study of the normability of the Lorentz spaces $L^{p,s}(R,\mu)$ goes back to the work of G.G. Lorentz [10, 11] (see also [13, 3, 2] for a more recent account of the normability results for the weighted Lorentz spaces). The condition defining these spaces is given in terms of the distribution function and, equivalently, the non-increasing rearrangement of f (see [1] for standard notations and basic definitions):

$$\|f\|_{p,s} = \left(\int_0^\infty (t^{1/p} f^*(t))^s \, \frac{dt}{t} \right)^{1/s},$$

with the usual modification if $s = \infty$. Lorentz proved that $\| \ \|_{p,s}$ is a norm, if and only if $1 \le s \le p < \infty$, and the space $L^{p,s}(R,\mu)$ is always normable (i.e., there exists a norm equivalent to $\| \ \|_{p,s}$), for the range $1 < p < s \le \infty$ (for the remaining cases it is known that $L^{p,s}(R,\mu)$

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cannot be endowed with an equivalent norm). From now on we will only consider the range $1 < p < \infty,\ 1 \leq s \leq \infty$.

Note that the spaces $L^{p,s}$, with $p < s$, play an important role not only as dual spaces for the Banach spaces $L^{p',s'}$ (see [1, 7]). For example, they arise naturally in limiting embeddings of Lipschitz spaces ([8]).

The study of the normability for $p < s$ was carried out by means of the *maximal norm:*

$$\|f\|^*_{p,s} = \left(\int_0^\infty (t^{1/p} f^{**}(t))^s \, \frac{dt}{t} \right)^{1/s},$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(x)\, dx.$$

It is easy to see that $\| \ \|^*_{p,s}$ is always a norm. Moreover, one can prove that $\| \ \|^*_{p,s}$ is equivalent to $\| \ \|_{p,s}$, with the following optimal estimates:

$$(p')^{1/s} \|f\|_{p,s} \leq \|f\|^*_{p,s} \leq p' \|f\|_{p,s} \tag{1.1}$$

(see [14, 9]; as usual, p' denotes the conjugate exponent, $1/p + 1/p' = 1$).

As a consequence of the fact that $\| \ \|_{p,s}$ is equivalent to a norm, it is easy to see that it is a quasi-norm satisfying the triangle inequality, uniformly on the number of terms: there exists a constant $c_{p,s} > 0$ such that, for every finite collection $\{f_k\}_{k=1,\cdots,N} \subset L^{p,s}(R, \mu)$:

$$\left\| \sum_{k=1}^N f_j \right\|_{p,s} \leq c_{p,s} \sum_{k=1}^N \|f_j\|_{p,s}. \tag{1.2}$$

It can readily be proved the converse result; namely, (1.2) is equivalent to the fact that $\| \ \|_{p,s}$ is normable and, even more, that an alternative equivalent norm is given by means of the following *decomposition norm:*

$$\|f\|_{(p,s)} = \inf \left\{ \sum_k ||f_k||_{p,s} \right\}, \tag{1.3}$$

where the infimum is taken over all finite representations $f = \sum_k f_k$.

It is easy to prove that $\| \ \|_{(p,s)}$ is a norm, equivalent to $\| \ \|_{p,s}$, that agrees with $\| \ \|_{p,s}$ if $1 \leq s \leq p$. Moreover, the best constant in the inequality $\|f\|_{p,s} \leq c_{p,s} \|f\|_{(p,s)}$ is the same as the optimal one in (1.2). One of the main problems studied in this paper is to find the best constant in the triangle inequality (1.2) and its continuous version, the Minkowski integral inequality (the control of these constants is sometimes very relevant for estimating different type of integral operators,

where the use of the maximal norm and the inequalities (1.1) do not usually give optimal results).

For the Lorentz norms we have the following version of Hölder's inequality: if $f \in L^{p,s}(R,\mu)$ and $g \in L^{p',s'}(R,\mu)$ $(1 < p < \infty,\ 1 \leq s \leq \infty)$, then

$$\left| \int_R fg\, d\mu \right| \leq \|f\|_{p,s} ||g||_{p',s'} \tag{1.4}$$

(see [1, p. 220]).

In the theory of Banach Function Spaces ($L^{p,s}(R,\mu)$ is the canonical example in this context), and based on (1.4), it is also very natural to consider another norm defined in terms of the Köthe duality, which is denoted as the *dual norm:*

$$||f||'_{p,s} = \sup \left\{ \int_R fg\, d\mu : ||g||_{p',s'} = 1 \right\}. \tag{1.5}$$

As in the case of the decomposition norm, $\|\ \|'_{p,s}$ is a norm, equivalent to $\|\ \|_{p,s}$ and $\|f\|'_{p,s} = \|f\|_{p,s}$, if $1 \leq s \leq p$ (see (4.5)). Therefore, $\|f\|'_{p,s} = \|f\|_{(p,s)}$ $(1 \leq s \leq p)$.

The main result that we will prove in this paper shows that the decomposition and dual norm agree in the whole range of indices (Theorem 5.2), in spite of their quite different definitions. We also find the best constants in the inequalities relating either of these norms and $\|\ \|_{p,s}$ (see (4.4), Theorem 4.4, and Remark 4.3). In particular, these results give an alternative proof of the normability of $L^{p,s}(R,\mu)$ with *optimal estimates*. We would like to remark that, while (1.1) follows easily from standard estimates, finding the best constants in our context requires new ideas and much more complicated constructions.

In Section 2 we prove several technical lemmas used in subsequent sections. Section 3 introduces one of the key tools used in the paper: the level function (see Theorems 3.1 and 3.2). Sections 4 and 5 are the core of the paper, dealing with both the dual and decomposition norms, and proving the main results already mentioned above. Finally, in Section 6 we obtain the best constant in both the triangle and Minkowski's integral inequalities for the Lorentz spaces.

Throughout this paper (R,μ) denotes a $\sigma-$finite nonatomic measure space.

2. Auxiliary propositions

In this section we consider some auxiliary results that will be used in the sequel. We begin with some general inequalities.

Lemma 2.1. *Let f and g be non-increasing nonnegative functions on $[0,1]$. Then*

$$\int_0^1 f(x)dx \int_0^1 g(x)dx \le \int_0^1 f(x)g(x)dx.$$

This is the classical Chebyshev inequality (see, e.g., [6]).

Corollary 2.2. *Let g be a non-increasing nonnegative function on $[0,1]$ and let $0 < \alpha < 1$. Then*

$$\int_0^1 g(x)dx \le (1-\alpha)\int_0^1 g(x)x^{-\alpha}dx. \tag{2.1}$$

Lemma 2.3. *Let $p, s \in (1,\infty)$. Then for any $t \in [0,1]$*

$$(1-t^{s/p})^{1/s}(1-t^{s'/p'})^{1/s'} \le 1-t. \tag{2.2}$$

Proof. We will prove that for all $x, y \in (0,1)$

$$(1-x^s)^{1/s}(1-y^{s'})^{1/s'} \le 1-xy. \tag{2.3}$$

Then (2.2) will follow from (2.3) if we take $x = t^{1/p}, y = t^{1/p'}$. To prove (2.3), fix y and denote

$$\varphi(x) = 1 - xy - (1-x^s)^{1/s}(1-y^{s'})^{1/s'}.$$

We have

$$\varphi^{'}(x) = -y - \frac{x^{s-1}}{(1-x^s)^{1/s'}}(1-y^{s'})^{1/s'}.$$

Set $\varphi^{'}(x) = 0$. Then

$$\frac{(1-x^s)^{1/s'}}{x^{s-1}} = \frac{(1-y^{s'})^{1/s'}}{y}$$

and

$$\left(\frac{1}{x^s} - 1\right)^{1/s'} = \left(\frac{1}{y^{s'}} - 1\right)^{1/s'}.$$

This implies that $x^s = y^{s'}$, and hence, the function φ has an absolute minimum for $x = y^{1/(s-1)}$ and this minimum is 0, which proves (2.3). $\square$

The following lemma gives the sharp constant in the relation between Lorentz norms with different second indices (see [14, p. 192]).

Lemma 2.4. *Let $1 \le p < \infty$ and $1 \le r < s \le \infty$. Then, for any function $f \in L^{p,r}(R,\mu)$*

$$\left(\frac{p}{s}\right)^{1/s} \|f\|_{p,s} \le \left(\frac{p}{r}\right)^{1/r} \|f\|_{p,r}. \tag{2.4}$$

We consider now some auxiliary statements related to dual norm and decomposition norm.

Lemma 2.5. *Let* $f \in L^{p,s}(R,\mu)$ $(1 < p < \infty, 1 \le s \le \infty)$. *Then*

$$||f||'_{p,s} = ||f^*||'_{p,s}. \tag{2.5}$$

The proof can be found in [1, p. 45-49].

Lemma 2.6. *Let* $f \in L^{p,s}(R,\mu)$ $(1 < p < \infty,\ 1 \le s \le \infty)$. *Then*

$$||f||'_{p,s} \le ||f||_{(p,s)}. \tag{2.6}$$

Proof. Let $g \in L^{p',s'}(R,\mu)$ and let

$$f = \sum_k f_k. \tag{2.7}$$

Then, by Hölder's inequality (1.4),

$$\int_R |fg| d\mu \le \sum_k \int_R |f_k g| d\mu \le ||g||_{p',s'} \sum_k ||f_k||_{p,s}.$$

Taking infimum over all representations (2.7), we obtain (2.6). □

We shall use the following properties of the decomposition norm.

Lemma 2.7. *Let* $f \in L^{p,s}(R,\mu)$ $(1 < p < \infty,\ 1 \le s \le \infty)$. *Then:*

(1) *the equality*

$$||f||_{(p,s)} = \inf\left\{\sum_k ||f_k||_{p,s}\right\}, \tag{2.8}$$

holds, where the infimum is taken over all finite sequences $\{f_k\}$ *such that* $f_k \ge 0$ *and*

$$|f(x)| = \sum_k f_k(x);$$

(2) *if* $0 \le g \le f$, *then* $||g||_{(p,s)} \le ||f||_{(p,s)}$;

(3) *if* $0 \le g_n \le f$ *and* $g_n(x) \uparrow f(x)$ $\mu-$*almost everywhere on* R, *then* $||g_n||_{(p,s)} \to ||f||_{(p,s)}$.

Proof. Denote by σ the right hand side of (2.8). We have $f = \sum_k g_k$, where $g_k = f_k \operatorname{sign} f$ and therefore

$$\|f\|_{(p,s)} \le \sum_k \|g_k\|_{p,s} = \sum_k \|f_k\|_{p,s}.$$

Thus, $\|f\|_{(p,s)} \le \sigma$. On the other hand, for any $\varepsilon > 0$ there exists a representation $f = \sum_k g_k$ such that

$$\|f\|_{(p,s)} > \sum_k \|g_k\|_{p,s} - \varepsilon.$$

We have $|f| \le \sum_k |g_k| \equiv G$. Set $f_k = |f g_k|/G$. Then $\|f_k\|_{p,s} \le \|g_k\|_{p,s}$ and $|f| = \sum_k f_k$. Thus $\|f\|_{(p,s)} \ge \sigma - \varepsilon$, which proves (2.8). Further, statement (2) follows immediately from statement (1). To prove (3), observe that $||f - g_n||_{p,s} \to 0$ (see [1, p. 41]). Since

$$||g_n||_{(p,s)} \le ||f||_{(p,s)} \le ||g_n||_{(p,s)} + ||f - g_n||_{p,s},$$

we obtain (3). □

Lemma 2.8. *For each $f \in L^{p,s}(R,\mu)$*

$$||f||_{(p,s)} \le ||f^*||_{(p,s)}. \tag{2.9}$$

Proof. It is known that there exists a measure preserving transformation $\sigma : R \to (0, \mu(R))$ such that

$$f(x) = f^*(\sigma(x)), \quad \mu\text{-a.e. on} \quad R, \tag{2.10}$$

(see [1, p. 82, 83]). Let $f^*(t) = \sum_{k=1}^N g_k(t),\ g_k \ge 0$. Then $f(x) = \sum_{k=1}^N g_k(\sigma(x))$. Since $g_k \circ \sigma$ and g_k are equimeasurable, we have that

$$\|f\|_{(p,s)} \le \sum_{k=1}^N \|g_k \circ \sigma\|_{p,s} = \sum_{k=1}^N \|g_k\|_{p,s}.$$

This implies (2.9). □

It will be proved below that for any f we have the equality in (2.9).

Lemma 2.9. *Let $1 < p < \infty$. Assume that $f \in L^{p,s_0}(R,\mu)$ for some $p \le s_0 < \infty$. Then*

$$||f||_{p,\infty} = \lim_{s\to\infty} ||f||_{p,s} \tag{2.11}$$

and

$$||f||_{(p,\infty)} = \lim_{s\to\infty} ||f||_{(p,s)}. \tag{2.12}$$

Proof. To prove (2.11), we can assume that $\mu(\operatorname{supp} f) < \infty$. Then (2.11) follows from a similar property for the L^s-norm (see [4, p. 226]).

We shall prove (2.12). By Lemma 2.7, we can assume that $f \ge 0$ and consider only representations

$$f = \sum_{k=1}^N f_k, \quad \text{where} \quad f_k \ge 0. \tag{2.13}$$

For an arbitrary representation (2.13) we have that, for any $s > s_0$

$$||f||_{(p,s)} \leq \sum_{k=1}^{N} ||f_k||_{p,s}.$$

By (2.11), we obtain that

$$\overline{\lim_{s\to\infty}}\, ||f||_{(p,s)} \leq \sum_{k=1}^{N} ||f_k||_{p,\infty}$$

which implies that

$$\overline{\lim_{s\to\infty}}\, ||f||_{(p,s)} \leq ||f||_{(p,\infty)}. \tag{2.14}$$

To prove the reverse inequality, take an arbitrary $\varepsilon > 0$. For a fixed $s > s_0$, find a decomposition (2.13) such that

$$||f||_{(p,s)} > \sum_{k=1}^{N} ||f_k||_{p,s} - \varepsilon.$$

Applying inequality (2.4), we obtain

$$\begin{aligned} ||f||_{(p,s)} &> \left(\frac{s}{p}\right)^{1/s} \sum_{k=1}^{N} ||f_k||_{p,\infty} - \varepsilon \\ &> \sum_{k=1}^{N} ||f_k||_{p,\infty} - \varepsilon \geq ||f||_{(p,\infty)} - \varepsilon. \end{aligned}$$

Thus, $||f||_{(p,s)} > ||f||_{(p,\infty)} - \varepsilon$, for any $s > \sigma_0$ and any $\varepsilon > 0$. It follows that

$$\underline{\lim}_{s\to\infty}\, ||f||_{(p,s)} \geq ||f||_{(p,\infty)},$$

which, together with (2.14), proves (2.12). $\square$

Lemma 2.10. *Let* $h(x) = \chi_{[0,1]}(x)$ *and* $1 < p < \infty$, $1 \leq s \leq \infty$. *Then*

$$||h||_{p,s} = \left(\frac{p}{s}\right)^{1/s}. \tag{2.15}$$

If $p < s$, *then*

$$||h||'_{p,s} = ||h||_{(p,s)} = \left(\frac{s'}{p'}\right)^{1/s'}. \tag{2.16}$$

Proof. The equality (2.15) is immediate. We shall prove (2.16). Denote $\alpha = 1 - s'/p'$ and set

$$\varphi(t) = (1-\alpha)t^{-\alpha}, t \in (0,1]. \tag{2.17}$$

We have

$$||\varphi\chi_{[0,1]}||_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'}. \tag{2.18}$$

To evaluate the dual norm of h, we assume that $g \in L^{p',s'}(\mathbb{R}_+)$, $g \geq 0$ and $\|g\|_{p',s'} = 1$. Applying (2.1), Hölder's inequality (1.4), and (2.18), we obtain

$$\begin{aligned}\int_{\mathbb{R}_+} h(x)g(x)dx &\leq \int_0^1 g^*(x)dx \\ &\leq (1-\alpha)\int_0^1 g^*(x)x^{-\alpha}dx \leq \|g\|_{p',s'}\|\varphi\|_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'}.\end{aligned}$$

On the other hand, if,

$$g(x) = \left(\frac{s'}{p'}\right)^{1/s'} \chi_{[0,1]}(x),$$

then $\|g\|_{p',s'} = 1$ and

$$\int_{\mathbb{R}_+} h(x)g(x)dx = \left(\frac{s'}{p'}\right)^{1/s'}.$$

Thus,

$$\|h\|'_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'}. \tag{2.19}$$

We prove now the second equality in (2.16) (in Section 4 we shall prove that the dual and the decomposition norms always agree, but the proof of this fact for a characteristic function is much simpler). Let $1 < p < s < \infty$. Assume that the function φ in (2.17) is extended to the whole line $\mathbb{R}$ periodically with period 1. Set

$$g_N(x) = N\int_x^{x+1/N} \varphi(t)dt.$$

Then $g_N(x) \to \varphi(x)$ as $N \to \infty$ for all $x \in (0,1)$. Moreover,

$$\left(g_N\chi_{[0,1]}\right)^*(t) \leq \left(g_N\chi_{[0,1]}\right)^{**}(t) \leq \varphi^{**}(t) = t^{-\alpha}, \quad t \in (0,1].$$

Applying Lebesgue's dominated convergence theorem and (2.18), we obtain

$$\|g_N\chi_{[0,1]}\|_{p,s} \to \|\varphi\chi_{[0,1]}\|_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'}.$$

Let $\varepsilon > 0$. Fix a number N such that

$$\|g_N \chi_{[0,1]}\|_{p,s} < \left(\frac{s'}{p'}\right)^{1/s'} + \varepsilon \tag{2.20}$$

Set

$$f_k(x) = \int_{(k-1)/N}^{k/N} \varphi(x+t)dt = \frac{1}{N} g_N\left(x + \frac{k-1}{N}\right), \quad k = 1, \dots, N.$$

Then

$$\sum_{k=1}^{N} f_k(x) \;=\; \int_0^1 \varphi(x+t)dt = 1, \tag{2.21}$$

for all x. Since f_k are 1-periodic and $f_k(x) = f_1\left(x + (k-1)/N\right)$, the restrictions of f_k to $[0,1]$ are pairwise equimeasurable. Set now

$$h_k(x) = f_k(x)\chi_{[0,1]}(x), \quad k = 1, \dots, N.$$

Then, by (2.21), $h = \sum_{k=1}^{N} h_k$ and by (2.20)

$$\sum_{k=1}^{N} \|h_k\|_{p,s} < \left(\frac{s'}{p'}\right)^{1/s'} + \varepsilon.$$

This implies that

$$\|h\|_{(p,s)} \le \left(\frac{s'}{p'}\right)^{1/s'}, \quad \text{for} \quad p < s < \infty. \tag{2.22}$$

By Lemma 2.9, (2.22) holds for all $p < s \le \infty$. The opposite inequality follows from (2.19) and Lemma 2.6. $\square$

We shall use the following Hardy's lemma [1, p. 56].

Lemma 2.11. *Let f_1 and f_2 be nonnegative measurable functions on $\mathbb{R}_+$, such that*

$$\int_0^t f_1(u)\,du \le \int_0^t f_2(u)\,du,$$

for all $t > 0$. Then, for every nonnegative and non-increasing function g on $\mathbb{R}_+$, we have that

$$\int_0^\infty f_1(u)g(u)\,du \le \int_0^\infty f_2(u)g(u)\,du.$$

Finally, we recall the definition of the Hardy-Littlewood-Pólya relation. Let (R,μ) be a measure space and let f and g be $\mu-$measurable and $\mu-$a.e. finite functions on R. We write $f \prec g$ if

$$\int_0^t f^*(u)\,du \leq \int_0^t g^*(u)\,du,$$

for all $t>0$ (see [1, p. 55]).

3. The level function

The notion of a level function was first introduced by Halperin [5]. We shall use the extension of this notion given by Lorentz [12] and based on the following theorem.

Theorem 3.1. *Let φ be a positive measurable function on $\mathbb{R}_+$ such that*

$$\Phi(t)=\int_0^t \varphi(u)\,du<\infty,$$

for all $t>0$. Assume that f is a nonnegative measurable function on $\mathbb{R}_+$ and that

$$\int_0^t f(u)du = o(\Phi(t)), \quad \text{as} \quad t\to\infty.$$

Then, there exists a nonnegative function f° on $\mathbb{R}_+$ satisfying the following conditions:

(a) *the function $f^\circ(t)/\varphi(t)$ decreases on $\mathbb{R}_+$;*
(b) *$f \prec f^\circ$;*
(c) *up to a set of measure zero, the set $\{t\in\mathbb{R}_+ : f(t)\neq f^\circ(t)\}$ is the union of bounded disjoint intervals I_k such that*

$$\int_{I_k} f(u)du = \int_{I_k} f^\circ(u)du,$$

and $f^\circ(t)/\varphi(t)$ is constant on I_k.

This theorem is a slight modification of the results in [5] and [12, §3.6]; the proof is similar to the one given in [12, §3.6] for functions defined on $[0,1]$. It is easy to show that the function f° is uniquely determined (see [5, Theorem 3.7]). It is called *the level function of f with respect to φ*.

Theorem 3.2. *Let $1<p<\infty$ and $p<s\leq\infty$. Suppose that $f\in L^{p,s}(\mathbb{R}_+)$ is a nonnegative and non-increasing function on $\mathbb{R}_+$. Let f° be the level function of f with respect to the function $\varphi_0(t)=t^{-\alpha},\ \alpha = 1-s'/p'$. Then*

$$||f^\circ||_{p,s} \leq ||f||_{p,s} \leq c_{p,s}||f^\circ||_{p,s}, \tag{3.1}$$

where

$$c_{p,s} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'}. \tag{3.2}$$

The constants in the inequalities (3.1) are optimal.

Proof. First we assume that $s < \infty$. We consider the left hand side inequality in (3.1). Applying Theorem 3.1(c), we have $f^\circ(t) = \lambda_k t^{-\alpha}$ for all $t \in I_k$, where

$$\lambda_k = \left(\int_{I_k} t^{-\alpha}\,dt\right)^{-1} \int_{I_k} f(t)\,dt.$$

Since $\alpha = (s/p-1)/(s-1)$, and $f^\circ(t)^{s-1}t^{s/p-1} = \lambda_k^{s-1}$ then, applying Hölder's inequality, we obtain

$$\begin{aligned}\int_{I_k} f^\circ(t)^s t^{s/p-1}\,dt = \lambda_k^{s-1}\int_{I_k} f^\circ(t)\,dt &= \left(\int_{I_k} t^{-\alpha}\,dt\right)^{1-s}\left(\int_{I_k} f(t)\,dt\right)^s\\ &\le \int_{I_k} f(t)^s t^{s/p-1}\,dt.\end{aligned} \tag{3.3}$$

This estimate and property (c) yield the first inequality in (3.1).

Now, denote

$$\psi(t) = f(t)^{s-1}t^{s/p-1}. \tag{3.4}$$

Let $\widetilde{\psi}(t)$ be the level function of ψ with respect to $\varphi(t) = 1$. Applying Theorem 3.1, Lemma 2.11, and the inequality (1.4), we obtain

$$\begin{aligned}||f||_{p,s}^s = \int_0^\infty f(t)\psi(t)\,dt &\le \int_0^\infty f(t)\widetilde{\psi}(t)\,dt\\ &\le \int_0^\infty f^\circ(t)\widetilde{\psi}(t)\,dt \le ||f^\circ||_{p,s}||\widetilde{\psi}||_{p',s'}.\end{aligned}$$

To obtain the second inequality in (3.1), it suffices to prove that

$$||\widetilde{\psi}||_{p',s'} \le c_{p,s}||f||_{p,s}^{s-1}, \tag{3.5}$$

where the constant $c_{p,s}$ is defined by (3.2).

Let $E = \{t \in \mathbb{R}_+ : \widetilde{\psi}(t) = \psi(t)\}$. Then, up to a set of measure zero,

$$\mathbb{R}_+ \setminus E = \bigcup_k (a_k, b_k),$$

where (a_k, b_k) are bounded disjoint intervals such that

$$\widetilde{\psi}(t) = \frac{1}{b_k - a_k}\int_{a_k}^{b_k} \psi(u)du, \quad \text{for all} \quad t \in (a_k, b_k). \tag{3.6}$$

By Hölder's inequality

$$\int_{a_k}^{b_k} \psi(u)du \le \left(\int_{a_k}^{b_k} u^{s/p-1}\,du\right)^{1/s} \left(\int_{a_k}^{b_k} f(u)^s u^{s/p-1}\,du\right)^{1/s'}$$
$$= \left(\frac{p}{s}\right)^{1/s} (b_k^{s/p} - a_k^{s/p})^{1/s} \left(\int_{a_k}^{b_k} f(u)^s u^{s/p-1}\,du\right)^{1/s'}.$$

Using (3.6) and applying Lemma 2.3, we obtain that

$$\widetilde{\psi}(t) \le \left(\frac{p}{s}\right)^{1/s} (b_k^{s'/p'} - a_k^{s'/p'})^{-1/s'} \left(\int_{a_k}^{b_k} f(u)^s u^{s/p-1}\,du\right)^{1/s'},$$

for all $t \in (a_k, b_k)$. Thus,

$$\int_{a_k}^{b_k} \widetilde{\psi}(t)^{s'} t^{s'/p'-1}\,dt \le \left(\frac{p}{s}\right)^{s'/s} (b_k^{s'/p'} - a_k^{s'/p'})^{-1}$$
$$\times \int_{a_k}^{b_k} f(t)^s t^{s/p-1}\,dt \int_{a_k}^{b_k} t^{s'/p'-1}\,dt$$
$$= \left(\frac{p}{s}\right)^{s'/s} \frac{p'}{s'} \int_{a_k}^{b_k} f(t)^s t^{s/p-1}\,dt.$$

We also have that

$$\int_E \widetilde{\psi}(t)^{s'} t^{s'/p'-1}\,dt = \int_E \psi(t)^{s'} t^{s'/p'-1}\,dt$$
$$= \int_E f(t)^s t^{s/p-1}\,dt.$$

Since

$$c_{p,s} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'} > 1,$$

we obtain (3.5). Thus, the inequalities in (3.1) are proved for $s < \infty$.

Let now $s = \infty$ and hence $\alpha = 1/p$. For any k,

$$p'(b_k^{1/p'} - a_k^{1/p'})\lambda_k = \int_{a_k}^{b_k} f^\circ(t)dt = \int_{a_k}^{b_k} f(t)dt$$
$$\le ||f||_{p,\infty} \int_{a_k}^{b_k} t^{-1/p}dt = p'(b_k^{1/p'} - a_k^{1/p'})||f||_{p,\infty}.$$

Thus, $\lambda_k \le ||f||_{p,\infty}$, which implies that $||f^\circ||_{p,\infty} \le ||f||_{p,\infty}$. On the other hand, for any $t \in (a_k, b_k)$ we have (see Theorem 3.1 (b))

$$t^{1/p} f(t) \le t^{1/p-1} \int_0^t f(u)du \le t^{1/p-1} \int_0^t f^\circ(u)du \le p'||f^\circ||_{p,\infty}.$$

This implies the second inequality in (3.1) for $s = \infty$.

The left hand side inequality in (3.1) becomes equality for $f(t) = t^{-\alpha}\chi_{[0,1]}(t)$. Further, let $f = \chi_{[0,1]}$. Then

$$||f||_{p,s} = \left(\frac{p}{s}\right)^{1/s}.$$

Next, $f^\circ(t) = (1-\alpha)t^{-\alpha}\chi_{[0,1]}(t)$,

$$||f^\circ||_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'},$$

and we have equality $||f||_{p,s} = c_{p,s}||f^\circ||_{p,s}$. Thus, the constants in (3.1) are optimal. □

Remark 3.3. Let $1 < p < s \leq \infty$. Let $f \in L^{p,s}(\mathbb{R}_+)$ be a nonnegative and non-increasing function on $\mathbb{R}_+$ and let f° be the level function of f with respect to the function $\varphi_\alpha(t) = t^{-\alpha}$ $(\alpha = 1 - s'/p')$. Then, the equality

$$||f^\circ||_{p,s} = ||f||_{p,s} \tag{3.7}$$

holds if and only if $f^\circ(t) = f(t)$, except for a countable set of points t. Indeed, the last inequality in (3.3) becomes equality if and only if $f(t)t^\alpha$ is constant on I_k.

In other words, (3.7) holds if and only if $f(t)t^\alpha$ decreases on $\mathbb{R}_+$.

4. The dual norm

Recall that for a function $f \in L^{p,s}(R,\mu)$ $(1 < p < \infty, 1 \leq s \leq \infty)$ its dual norm is defined by

$$||f||'_{p,s} = \sup\left\{\int_R fg\,d\mu : \quad ||g||_{p',s'} = 1\right\}, \tag{4.1}$$

where the supremum is taken over all functions $g \in L^{p',s'}(R,\mu)$ with $||g||_{p',s'} = 1$.

By Lemma 2.5 and the Hardy-Littlewood inequality [1, p. 44], for any function $f \in L^{p,s}(R,\mu)$ $(1 < p < \infty, 1 \leq s \leq \infty)$

$$||f||'_{p,s} = \sup\left\{\int_0^\infty f^*(t)g(t)\,dt : ||g||_{p',s'} = 1\right\}, \tag{4.2}$$

where the supremum is taken over all nonnegative and nonincreasing functions $g \in L^{p',s'}(\mathbb{R}_+)$ with $||g||_{p',s'} = 1$.

Suppose that $1 < p < \infty$ and $1 \leq s \leq \infty$. Let $f \in L^{p,s}(\mathbb{R}_+)$ and let $g \in L^{p',s'}(\mathbb{R}_+)$. By Hölder's inequality (1.4)

$$\int_0^\infty |f(t)g(t)|\,dt \leq ||f||_{p,s}||g||_{p',s'}. \tag{4.3}$$

It follows that

$$||f||'_{p,s} \le ||f||_{p,s}. \tag{4.4}$$

If $s \le p$, then we have the equality of norms

$$||f||'_{p,s} = ||f||_{p,s}. \tag{4.5}$$

Indeed,

$$||f||_{p,s}^s = \int_0^\infty f^*(t)\psi(t)\,dt, \quad \psi(t) = f^*(t)^{s-1}t^{s/p-1}.$$

If $s \le p$, then the function ψ is non-increasing and we have

$$||\psi||_{p',s'}^{s'} = \int_0^\infty \psi(t)^{s'} t^{s'/p'-1}\,dt = ||f||_{p,s}^s.$$

The latter two equalities imply that $||f||'_{p,s} \ge ||f||_{p,s}$. Together with (4.4) this yields (4.5). Observe also that the supremum in (4.2) is attained on the function $g(t) = \psi(t)/||\psi||_{p',s'}$.

Now we assume that $p < s \le \infty$. Let $f \in L^{p,s}(\mathbb{R}_+)$. If the function $f^*(t)t^{1-s'/p'}$ is non-increasing, then as above we have the equality (4.5). Let f be an arbitrary nonnegative function in $L^{p,s}(\mathbb{R}_+)$ and let $g \in L^{p',s'}(\mathbb{R}_+)$, $g \ge 0$, be a nonincreasing function. By Lemma 2.11, we have that

$$\int_0^\infty f(t)g(t)\,dt \le \inf_{f \prec h} ||h||_{p,s}||g||_{p',s'}. \tag{4.6}$$

This implies that

$$||f||'_{p,s} \le \inf_{f \prec h} ||h||_{p,s}. \tag{4.7}$$

Note that in the case $s \le p$ the infimum in (4.7) is equal to $||f||_{p,s}$. However, for $s > p$ the infimum may be smaller than $||f||_{p,s}$ and (4.6) may give a refinement of the inequality (4.3). It was proved by Halperin [5, Theorem 4.2] (see also [12, Theorem 3.6.5]) that equality in (4.7) holds and the infimum is attained for some $h \in L^{p,s}(\mathbb{R}_+)$. Since the proofs given in [5] and [12] do not cover explicitly the case $s = \infty$, and for the sake of completeness, we show the result for all $p < s \le \infty$.

Theorem 4.1. *Let $1 < p < s \le \infty$. Assume that $f \in L^{p,s}(\mathbb{R}_+)$ is a nonnegative and non-increasing function on $\mathbb{R}_+$. Set $\alpha = 1 - s'/p'$ and $\varphi_\alpha(t) = t^{-\alpha}$. Then*

$$||f||'_{p,s} = \inf_{f \prec h} ||h||_{p,s} = ||f^\circ||_{p,s}, \tag{4.8}$$

where f° is the level function of f with respect to the function φ_α.

Proof. In view of (4.7) and Theorem 3.1 (b), it suffices to prove that

$$||f||'_{p,s} \geq ||f^\circ||_{p,s}. \tag{4.9}$$

Set

$$E = \{x \in \mathbb{R}_+ : f(x) = f^\circ(x)\}.$$

By Theorem 3.1, up to a set of measure zero,

$$\mathbb{R}^+ \setminus E = \bigcup_k (a_k, b_k),$$

where (a_k, b_k) are disjoint bounded intervals such that

$$\int_{a_k}^{b_k} f(t)\,dt = \int_{a_k}^{b_k} f^\circ(t)\,dt. \tag{4.10}$$

We first assume that $s < \infty$. Denote $\psi(t) = f^\circ(t)^{s-1}t^{s/p-1}$. As above, we have

$$||\psi||^{s'}_{p',s'} = \int_0^\infty \psi(t)^{s'} t^{s'/p'-1}\,dt = ||f^\circ||^s_{p,s}.$$

Set $g(t) = \psi(t)/||f^\circ||^{s-1}_{p,s}$. Then $||g||_{p',s'} = 1$. For each k, we have $f^\circ(t) = \lambda_k t^{-\alpha}$ and $\psi(t) = \lambda_k^{s-1}$, for $t \in (a_k, b_k)$ (where λ_k is a constant). Thus,

$$\begin{aligned} ||f^\circ||^{s-1}_{p,s} \int_{a_k}^{b_k} f(t)g(t)dt &= \lambda_k^{s-1} \int_{a_k}^{b_k} f(t)dt \\ &= \lambda_k^{s-1} \int_{a_k}^{b_k} f^\circ(t)dt = \int_{a_k}^{b_k} [t^{1/p} f^\circ(t)]^s \frac{dt}{t}. \end{aligned}$$

Besides, we have

$$||f^\circ||^{s-1}_{p,s} \int_E f(t)g(t)dt = \int_E [t^{1/p} f^\circ(t)]^s \frac{dt}{t},$$

and thus,

$$\int_0^\infty f(t)g(t)dt = ||f^\circ||_{p,s},$$

from which we obtain (4.9).

Let now $s = \infty$. In this case we have

$$||f^\circ||_{p,\infty} = \lim_{t \to 0+} f^\circ(t)t^{1/p}. \tag{4.11}$$

We assume first that for some k we have $a_k = 0$. Set

$$g(t) = \chi_{(0,b_k)}/(p' b_k^{1/p'}).$$

Then $||g||_{p',1} = 1$. We have

$$f^\circ(t) = \lambda_k t^{-1/p} \quad \text{for} \quad t \in (0, b_k) \quad \text{and} \quad ||f^\circ||_{p,\infty} = \lambda_k.$$

Thus,

$$\int_0^\infty f(t)g(t)dt = (p'b_k^{1/p'})^{-1}\int_0^{b_k} f(t)dt$$
$$= (p'b_k^{1/p'})^{-1}\int_0^{b_k} f^\circ(t)dt = \lambda_k = ||f^\circ||_{p,\infty}.$$

This implies (4.9).

Now we assume that $a_k \neq 0$ for each k. Then, for any $\delta > 0$ we have

$$(0,\delta)\cap A \neq \emptyset, \quad \text{where} \quad A = \mathbb{R}_+ \setminus \cup_j (a_j, b_j). \tag{4.12}$$

On the other hand, by Theorem 3.1 (c), for any $t \in A$

$$\int_0^t f(u)du = \int_0^t f^\circ(u)du. \tag{4.13}$$

Let $\varepsilon > 0$. By (4.11), there exists $\delta > 0$ such that

$$f^\circ(t)t^{1/p} > ||f^\circ||_{p,\infty} - \varepsilon \quad \text{for any} \quad t \in (0,\delta).$$

Let $\xi \in (0,\delta)\cap A$. Set $g(t) = \chi_{(0,\xi)}/(p'\xi^{1/p'})$. Then $||g||_{p',1} = 1$. Applying (4.13) and (4.12), we get

$$\int_0^\infty f(t)g(t)dt = (p'\xi^{1/p'})^{-1}\int_0^\xi f^\circ(t)dt > ||f^\circ||_{p,\infty} - \varepsilon,$$

which again implies (4.9). □

Remark 4.2. Note that for $1 < p < s < \infty$ the supremum in (4.2) is attained on the function $g(t) = \psi(t)/||\psi||_{p',s'}$, where $\psi(t) = f^\circ(t)^{s-1}t^{s/p-1}$. If $s = \infty$, then the supremum in (4.2) may not be attained.

Remark 4.3. Let $1 < p < s \leq \infty$, and let $f \in L^{p,s}(\mathbb{R}_+)$ be a non-negative and non-increasing function on $\mathbb{R}_+$. Then, by Remark 3.3, the equality

$$||f||'_{p,s} = ||f||_{p,s}$$

holds if and only if $f(t)t^\alpha$ decreases on $\mathbb{R}_+$.

The following theorem gives the sharp estimate of the standard norm via the dual norm.

Theorem 4.4. *Let $1 < p < \infty$ and $p < s \leq \infty$. Then, for any function $f \in L^{p,s}(R,\mu)$*

$$||f||_{p,s} \leq c_{p,s}||f||'_{p,s}, \tag{4.14}$$

where

$$c_{p,s} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'}.$$

The constant $c_{p,s}$ is optimal.

This theorem follows immediately from Theorems 4.1 and 3.2. However, a direct proof can be given exactly as in Theorem 3.2. Indeed, assume that f is nonnegative and non-increasing on $\mathbb{R}_+$. As in the proof of Theorem 3.2, we have

$$\begin{aligned} ||f||_{p,s}^s &= \int_0^\infty f(t)\psi(t)\,dt \le \int_0^\infty f(t)\widetilde{\psi}(t)\,dt \\ &\le ||f||'_{p,s}||\widetilde{\psi}||_{p',s'}. \end{aligned}$$

Applying the inequality (3.5), we obtain (4.14). Let now $f = \chi_{[0,1]}$. Then, by Lemma 2.10

$$||f||'_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'} \quad \text{and} \quad ||f||_{p,s} = \left(\frac{p}{s}\right)^{1/s} = c_{p,s}||f||'_{p,s},$$

which shows that the constant in (4.14) is optimal.

5. The decomposition norm

In this section we prove one of the main results of this paper –the coincidence of the dual and the decomposition norms. The following lemma plays an important role in the proof of the equality of these two norms.

Lemma 5.1. *Let $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_\nu$ be positive numbers and let $\{\eta_{jk}\}$ be a $(N \times \nu)-$matrix of positive numbers $(1 \le j \le N,\ 1 \le k \le \nu)$. Set*

$$\beta_k = \sum_{j=1}^N \eta_{jk}, \qquad k = 1, \cdots, \nu.$$

Assume that

$$\beta_1 + \cdots + \beta_k \ge \alpha_1 + \cdots + \alpha_k, \tag{5.1}$$

for any $k = 1, \ldots, \nu$. Let $\eta = \max \eta_{jk}$. Then, for any $j = 1, \ldots, N$ there exists a permutation $\{\tilde{\eta}_{jk}\}_{k=1}^\nu$ of the $\nu-$tuple $\{\eta_{jk}\}_{k=1}^\nu$ such that

$$\alpha_k \le \tilde{\beta}_k + \eta, \qquad \tilde{\beta}_k = \sum_{j=1}^N \tilde{\eta}_{jk}, \tag{5.2}$$

for any $k = 1, \ldots, \nu$.

Proof. For $\nu = 1$ the lemma is obvious. Assume that it is true for $\nu - 1$ ($\nu \geq 2$). We have $\beta_1 \geq \alpha_1$. If $\beta_k \geq \alpha_1$ for all $k = 1, \ldots, \nu$, there is nothing to prove. Otherwise, denote by s the least natural k for which $\beta_k < \alpha_1$. Then, $s \geq 2$. Set $\gamma_0 = \beta_1$, $\gamma_N = \beta_s$, and

$$\gamma_m = \sum_{j=1}^{m} \eta_{js} + \sum_{j=m+1}^{N} \eta_{j1}, \quad \text{for} \quad 1 \leq m < N.$$

We have $\gamma_0 \geq \alpha_1$ and $\gamma_N < \alpha_1$. Let m_0 be the least m for which $\gamma_m < \alpha_1$. Since $|\gamma_m - \gamma_{m-1}| \leq \eta$ for any $m = 1, \cdots, N$, we have that

$$\gamma_{m_0} < \alpha_1 \leq \gamma_{m_0 - 1} \leq \gamma_{m_0} + \eta. \tag{5.3}$$

Set

$$\tilde{\eta}_{j1} = \begin{cases} \eta_{js} & \text{if } 1 \leq j \leq m_0 \\ \eta_{j1} & \text{if } m_0 < j \leq N, \end{cases}$$

$$\eta'_{js} = \begin{cases} \eta_{j1} & \text{if } 1 \leq j \leq m_0 \\ \eta_{js} & \text{if } m_0 + 1 < j \leq N, \end{cases}$$

and $\eta'_{jk} = \eta_{jk}$ $(j = 1, ..., N)$, if $k \neq 1, s$. Using (5.3) we have

$$\tilde{\beta}_1 < \alpha_1 \leq \tilde{\beta}_1 + \eta, \quad \text{where} \quad \tilde{\beta}_1 = \gamma_{m_0} = \sum_{j=1}^{N} \eta'_{js}. \tag{5.4}$$

Denote also $\beta'_k = \beta_k, \quad k = 2, ..., N$. We first assume that $s = 2$. We have

$$\beta_1 + \beta_2 + \cdots + \beta_k \geq \alpha_1 + \alpha_2 + \cdots + \alpha_k,$$

for each $k \geq 2$. But $\beta_1 + \beta_2 = \tilde{\beta}_1 + \beta'_2$ and $\tilde{\beta}_1 < \alpha_1$, by (5.4). Thus,

$$\beta'_2 + \cdots + \beta'_k \geq \alpha_2 + \cdots + \alpha_k, \qquad k = 2, \ldots, \nu.$$

Now we assume that $s > 2$. Then we have for every $2 \leq l < s$

$$\beta'_l = \beta_l \geq \alpha_1 \geq \alpha_l,$$

and therefore

$$\beta'_2 + \cdots + \beta'_k \geq \alpha_2 + \cdots + \alpha_k, \qquad 2 \leq k < s.$$

Let $k \geq s$. Since

$$\tilde{\beta}_1 + \beta'_2 + \cdots + \beta'_k = \beta_1 + \beta_2 + \cdots + \beta_k,$$

and $\tilde{\beta}_1 < \alpha_1$ (see (5.4)), it follows from (5.1) that

$$\beta'_2 + \cdots + \beta'_k \geq \alpha_2 + \cdots + \alpha_k.$$

Thus, we can apply our inductive assumption to the $(N \times (\nu - 1))$-matrix

$$\{\eta'_{jk}\}, \quad 1 \leq j \leq N, \; 2 \leq k \leq \nu.$$

Together with (5.4), this proves the lemma. □

Theorem 5.2. *Let $1 < p < \infty$ and $1 \le s \le \infty$. Then, for any function $f \in L^{p,s}(R,\mu)$*
$$||f||'_{p,s} = ||f||_{(p,s)}. \tag{5.5}$$

Proof. If $s \le p$, then
$$||f||'_{p,s} = ||f||_{p,s} = ||f||_{(p,s)}.$$
We assume that $1 < p < s \le \infty$. By Lemma 2.6,
$$||f||'_{p,s} \le ||f||_{(p,s)}.$$
We shall prove that
$$||f||_{(p,s)} \le ||f||'_{p,s}. \tag{5.6}$$

By virtue of (2.5) and (2.9), it suffices to prove (5.6) in the case when (R,μ) is $\mathbb{R}_+$, with Lebesgue's measure, and f is a nonnegative and non-increasing function on $\mathbb{R}_+$. Applying Lemma 2.7(3), we can also assume that there exist $0 < x_0 < x_1 < \infty$ such that $f(x) = c_0 > 0$ on $(0, x_0)$ and $f(x) = 0$ for all $x > x_1$.

By Theorem 4.1,
$$||f||'_{p,s} = ||f^\circ||_{p,s},$$
where f° is the level function of f with respect to the function $\varphi_\alpha(t) = t^{-\alpha},\ \alpha = 1 - s'/p'$. Set
$$E = \{x \in \mathbb{R}_+ : f(x) = f^\circ(x)\}.$$
By Theorem 3.1, up to a set of measure zero,
$$\mathbb{R}^+ \setminus E = \bigcup_i (a_i, b_i),$$
where (a_i, b_i) are disjoint bounded intervals such that
$$\int_{a_i}^x f(t)\,dt \le \int_{a_i}^x f^\circ(t)\,dt, \qquad x \in (a_i, b_i) \tag{5.7}$$
and
$$\int_{a_i}^{b_i} f(t)\,dt = \int_{a_i}^{b_i} f^\circ(t)\,dt. \tag{5.8}$$

By our assumption, $f(x) = c_0$ on $(0, x_0)$. At the same time, f° is strictly decreasing on $(0, x_0)$. This implies that, for some i we have $a_i = 0$. Indeed, assume the contrary. Then, as is easily seen, there exists (a_k, b_k) such that $0 < a_k < b_k < x_0$. We have $f^\circ(x) = \lambda_k x^{-\alpha}$ on (a_k, b_k). Further,
$$c_0 a_k = \int_0^{a_k} f^\circ(x)dx > \frac{\lambda_k a_k^{1-\alpha}}{1-\alpha},$$

and therefore

$$\lambda_k < c_0(1-\alpha)a_k^\alpha.$$

From here,

$$\begin{aligned} c_0 b_k &= \int_0^{b_k} f^\circ(x)dx = c_0 a_k + \int_{a_k}^{b_k} f^\circ(x)dx \\ &= c_0 a_k + \frac{\lambda_k}{1-\alpha}(b_k^{1-\alpha} - a_k^{1-\alpha}) < c_0 b_k. \end{aligned}$$

Thus, we can assume that $a_1 = 0$. Let $b = \max(x_1, \sup_j b_j)$. Then $b < \infty$ and $f^\circ(x) = 0$ for all $x > b$.

Let $\varepsilon > 0$. For any $\nu \in \mathbb{N}$, define the function g_ν in the following way. First, set $g_\nu(x) = f(x)$ for $x \in E$; then $g_\nu(x) = 0$ for all $x > b$. Further, we subdivide each interval (a_i, b_i) into ν subintervals Δ_k^i, $k = 1, \cdots, \nu$, of length $|\Delta_k^i| = (b_i - a_i)/\nu$, and set

$$g_\nu(x) = |\Delta_k^i|^{-1} \int_{\Delta_k^i} f(t)\,dt \quad \text{for} \quad x \in \Delta_k^i, \quad k = 1, ..., \nu.$$

It is easy to see that there exists ν_1 such that

$$\|f - g_\nu\|_{p,s} < \varepsilon, \tag{5.9}$$

for all $\nu \geq \nu_1$. It follows that, for all $\nu \geq \nu_1$,

$$\begin{aligned} \|f\|_{(p,s)} &\leq \|g_\nu\|_{(p,s)} + \|f - g_\nu\|_{p,s} \\ &\leq \|g_\nu\|_{(p,s)} + \varepsilon. \end{aligned} \tag{5.10}$$

Similarly, for every $\nu \in \mathbb{N}$ we define the function ψ_ν approximating f°. Set $\psi_\nu(x) = f^\circ(x)$ for $x \in E$ and

$$\psi_\nu(x) = |\Delta_k^i|^{-1} \int_{\Delta_k^i} f^\circ(t)\,dt, \quad \text{for} \quad x \in \Delta_k^i, \quad k = 1, ..., \nu.$$

There exists an integer $\nu_2 \geq \nu_1$ such that

$$\|\psi_\nu\|_{p,s} \leq \|f^\circ\|_{p,s} + \varepsilon, \tag{5.11}$$

for all $\nu \geq \nu_2$. Fix $\nu \geq \nu_2$. Next, choose a number $\delta > 0$ such that

$$\delta < \varepsilon b^{-1/p} \left(\frac{s}{p}\right)^{1/s}. \tag{5.12}$$

We shall prove that there exist a number $N \in \mathbb{N}$ and functions $f_j \geq 0$, $j = 1, \ldots, N$, such that

$$g_\nu(x) \leq \sum_{j=1}^N f_j(x) + \delta, \qquad x > 0, \tag{5.13}$$

and

$$\|f_j\|_{p,s} = \|\psi_\nu\|_{p,s}/N, \; j = 1, \dots, N. \tag{5.14}$$

For any i, denote

$$\beta_k^{(i)} = |\Delta_k^i|^{-1} \int_{\Delta_k^i} f^\circ(t)\, dt, \; k = 1, ..., \nu.$$

There exists a number $N' \in \mathbb{N}$ such that

$$\beta_k^{(1)} < N'\delta, \qquad k = 1, \dots, \nu.$$

On the other hand, since f° is bounded on $[b_1, \infty)$, there exists $N'' \in \mathbb{N}$ such that, for all $i \geq 2$,

$$\beta_k^{(i)} < N''\delta, \qquad k = 1, \dots, \nu.$$

Let $N = \max(N', N'')$. Then, for any i

$$\beta_k^{(i)} < N\delta, \qquad k = 1, \dots, \nu.$$

Now we define the functions f_j, $j = 1, \dots, N$. Set

$$f_j(x) = \frac{1}{N}\psi_\nu(x), \quad \text{for} \quad x \in E.$$

Further, consider an interval (a_i, b_i). Set

$$\eta_{jk}^{(i)} = \frac{\beta_k^{(i)}}{N}, \quad j = 1, \dots, N, \quad k = 1, \dots, \nu.$$

Let

$$\alpha_k^{(i)} = |\Delta_k^i|^{-1} \int_{\Delta_k^i} f(t) dt.$$

Then, by (5.7)

$$\beta_1^{(i)} + \cdots + \beta_k^{(i)} \geq \alpha_1^{(i)} + \cdots + \alpha_k^{(i)}, \quad k = 1, \dots, \nu.$$

Applying Lemma 5.1, we obtain that, for any fixed i and every $j = 1, \dots, N$, there exists a permutation $\left\{\tilde{\eta}_{jk}^{(i)}\right\}_{k=1}^{\nu}$ of the ν-tuple $\left\{\eta_{jk}^{(i)}\right\}_{k=1}^{\nu}$ such that

$$\alpha_k^{(i)} \leq \sum_{j=1}^{N} \tilde{\eta}_{jk}^{(i)} + \delta, \tag{5.15}$$

for any $k = 1, \dots, \nu$. Set now

$$f_j(x) = \tilde{\eta}_{jk}^{(i)}, \quad \text{for} \quad x \in \Delta_k^i.$$

The functions f_j $(j = 1, \dots, N)$ are defined on $\mathbb{R}_+$ and each of them is equimeasurable with ψ_ν/N. Thus, we have (5.14). Moreover, (5.15)

implies (5.13). Applying (5.12)–(5.14), and taking into account that $g_\nu(x) = 0$ for $x > b$, we obtain

$$\|g_\nu\|_{(p,s)} \le \sum_{j=1}^{N} \|f_j\|_{p,s} + \delta||\chi_{[0,b]}||_{p,s} \le \|\psi_\nu\|_{p,s} + 2\varepsilon.$$

Using (5.10) and (5.11), we obtain

$$\|f\|_{(p,s)} \le \|f^\circ\|_{p,s} + 4\varepsilon.$$

This implies (5.6). $\square$

Corollary 5.3. *Let* $f \in L^{p,s}(R,\mu)$ $(1 \le p < \infty,\ 1 \le s \le \infty)$. *Then*

$$||f||_{(p,s)} = ||f^*||_{(p,s)}. \tag{5.16}$$

Indeed, (5.16) follows immediately from (2.5) and (5.5). Observe that (5.16) does not follow directly from the definition.

6. The triangle inequality

Applying Theorem 4.4 and Theorem 5.2, we immediately obtain the following version of the "triangle inequality."

Theorem 6.1. *Let* $1 < p < s \le \infty$. *Assume that* $f_k \in L^{p,s}(R,\mu)$ $(k = 1, ..., N)$. *Then*

$$\left\| \sum_{k=1}^{N} f_k \right\|_{p,s} \le c_{p,s} \sum_{k=1}^{N} ||f_k||_{p,s}, \tag{6.1}$$

where

$$c_{p,s} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'},$$

and the constant is optimal.

Remark 6.2. It is clear that (6.1) is equivalent to the inequality

$$||f||_{p,s} \le c_{p,s}||f||_{(p,s)}, \tag{6.2}$$

where f is any function in $L^{p,s}(R,\mu)$. Inequality (6.2) follows directly from (4.14) and Lemma 2.6. By Lemma 2.10, (6.2) becomes equality for $f = \chi_{[0,1]}$. Thus, Theorem 6.1 follows from Theorem 4.4 and Lemmas 2.6 and 2.10.

We also have the following continuous version of the Minkowski type inequality.

Theorem 6.3. *Suppose that* (R, μ) *is a* σ*-finite nonatomic measure space and* (Q, ν) *is a* σ*-finite measure space. Let* f *be a nonnegative measurable function on* $(R \times Q, \mu \times \nu)$*. Assume that* $1 < p < s \leq \infty$ *and that, for almost all* $y \in Q$ *the function*

$$f_y(x) = f(x, y), \quad x \in R,$$

belongs to $L^{p,s}(R, \mu)$*. Set* $F(x) = \int_Q f(x, y)d\nu(y)$*,* $x \in R$*. Then*

$$||F||_{p,s} \leq c_{p,s} \int_Q ||f_y||_{p,s} d\nu(y), \tag{6.3}$$

where

$$c_{p,s} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'}, \tag{6.4}$$

and the constant is optimal.

Proof. By Theorem 4.4,

$$||f||_{p,s} \leq c_{p,s} ||f||'_{p,s}, \tag{6.5}$$

where the constant $c_{p,s}$ is defined by (6.4). Let $g \in L^{p',s'}(R, \mu)$ and assume that $||g||_{p',s'} = 1$. Applying Fubini's Theorem and Hölder's inequality, we obtain

$$\begin{aligned} \int_R F(x)g(x)d\mu(x) &= \int_R \left(\int_Q f(x, y)d\nu(y)\right) g(x)d\mu(x) \\ &= \int_Q \int_R f(x, y)g(x)d\mu(x)d\nu(y) \leq \int_Q ||f_y||_{p,s} d\nu(y). \end{aligned}$$

Together with (6.5), this implies (6.3). Finally, it follows from Theorem 6.1 that the constant $c_{p,s}$ in (6.3) cannot be replaced by a smaller one. □

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